Asymptotic form of the density profile for Gaussian and Laguerre random matrix ensembles with orthogonal and symplectic symmetry

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In a recent study we have obtained correction terms to the large N asymptotic expansions of the eigenvalue density for the Gaussian unitary and Laguerre unitary ensembles of random $N \times N$ matrices, both in the bulk and at the soft edge of the spectrum. In the present study these results are used to similarly analyze the eigenvalue density for Gaussian and Laguerre random matrix ensembles with orthogonal and symplectic symmetry. As in the case of unitary symmetry, a matching is exhibited between the asymptotic expansion of the bulk density, expanded about the edge, and the asymptotic expansion of the edge density, expanded into the bulk. In addition, aspects of the asymptotic expansion of the smoothed density, which involves delta functions at the endpoints of the support, are interpreted microscopically.

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1 Introduction

Perhaps the best known result in random matrix theory is the Wigner semi-circle law (see e.g. [4]). Consider a real symmetric matrix, with elements on the diagonal i.i.d. random variables having finite variance and similarly the elements above the diagonal. The Wigner semi-circle law tells us that after appropriate scaling, the limiting eigenvalue density is given by the semi-circle functional form

$$\rho_{\rm W}(\lambda) = \begin{cases} \frac{2}{\pi}(1-\lambda^2)^{1/2}, & |\lambda| < 1 \\ 0, & |\lambda| \geq 1. \end{cases} \tag{1.1}$$

As a concrete example, the Gaussian orthogonal ensemble (GOE) of real symmetric matrices is specified by its diagonal entries being distributed according to the normal distribution $\mathrm{N}[0,1]$ and its upper triangular entries according to $\mathrm{N}[0,1/\sqrt{2}]$. Let $\rho^{(N)}(\lambda)$ denote the eigenvalue density for $N \times N$ matrices from the GOE. After the scaling $\sqrt{2N}\rho^{(N)}(\sqrt{2N}\lambda) \mapsto N\rho(\lambda)$ the $N \to \infty$ limiting form of $\rho^{(N)}(\lambda)$ is given by (1.1).

The functional form (1.1) has implications with respect to averaging a so called linear statistic $A = \sum_{j=1}^N a(\lambda_j)$ over the spectrum of random real symmetric matrices. Thus, if the $N \to \infty$ scaling is such that $\frac{\alpha_N}{N}\rho^{(N)}(\alpha_N \lambda) \to \rho_{\rm W}(\lambda)$, $a(\alpha_N\lambda) \to \tilde{a}(\lambda)$ for some α_N then

$$\langle A \rangle := \int_{-\infty}^{\infty} \rho^{(N)}(\lambda) a(\lambda)\, d\lambda \;\sim\; N \int_{-1}^{1} \rho_{\rm W}(\lambda)\tilde{a}(\lambda)\, d\lambda. \tag{1.2}$$

The result (1.2) immediately draws attention to corrections to the Wigner semi-circle law. Indeed in studies of the full distribution of linear statistics averaged over the GOE and certain of its

generalizations, it is necessary to compute the O(1) term in the asymptotic expansion of (1.2) [9]. For this one seeks the asymptotic expansion of $\bar{\rho}^{(N)}(\lambda)$, where $\bar{\rho}^{(N)}(\lambda)$ is the signed measure (smoothed density) such that

$$\int_{-\infty}^{\infty} \rho^{(N)}(\lambda) a(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \bar{\rho}^{(N)}(\lambda) a(\lambda)\, d\lambda$$

to all orders in the corresponding asymptotic expansions. In the case of the GOE itself the O(1) term is known [17, 2, 9], and one has

$$\frac{\sqrt{2N}}{N} \bar{\rho}^{(N)}(\sqrt{2N}\lambda) \;\sim\; \rho_{\mathrm{W}}(\lambda) + \frac{1}{N}\left(\frac{1}{4}\Big(\delta(\lambda-1)+\delta(\lambda+1)\Big) - \frac{1}{2\pi}\frac{1}{\sqrt{1-\lambda^2}}\chi_{|\lambda|<1}\right) \tag{1.3}$$

where $\chi_T = 1$ if T is true and $\chi_T = 0$ otherwise.

The correction term in (1.3) exhibits a most remarkable feature, namely delta functions at the edge of the support of the spectrum. The appearance of the delta functions at a microscopic level, when one seeks directly the asymptotic expansion of $\frac{\sqrt{2N}}{N}\rho^{(N)}(\sqrt{2N}\lambda)$ rather than the asymptotics of the smoothed quantity $\bar{\rho}^{(N)}(\lambda)$ has not, to the best of our knowledge, been previously studied. One of the purposes of this paper is to undertake such a study for the Gaussian and Laguerre ensembles in random matrix theory. Each of the three symmetry classes, orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) will be considered. For the Gaussian ensemble it is known [9] that (1.3) then generalizes to read

$$\frac{\sqrt{2N}}{N} \bar{\rho}^{(N)}(\sqrt{2N}\lambda) \;\sim\; \rho_{\mathrm{W}}(\lambda) + \frac{1}{N}\left(\frac{1}{\beta}-\frac{1}{2}\right)\left(\frac{1}{2}\Big(\delta(\lambda-1)+\delta(\lambda+1)\Big) - \frac{1}{\pi}\frac{1}{\sqrt{1-\lambda^2}}\chi_{|\lambda|<1}\right). \tag{1.4}$$

Our task is to relate this expansion to the asymptotic expansion of the density itself.

The expansion (1.4) clearly shows both a bulk effect and an edge effect. This is in keeping with there being both a (global) bulk regime, and an edge regime which must be treated separately in the asymptotic analysis. As these expansions relate to the same quantity, one would expect there to be a matching in an appropriate limit. This topic, initiated in [8] for the GUE and LUE, is another main theme of the present work.

We begin in Section 2 by recalling the results from [8] relating to the asymptotic expansions of the global bulk density, and the soft edge density, in the GUE and LUE. We then proceed to write down higher order terms in these asymptotic expansions (obtained from the method of [8]). These higher order terms are then used to further investigate the matching phenomenon alluded to above.

In Section 3 formulas required in the study of the asymptotics of the density in the Gaussian and Laguerre ensembles with orthogonal and symplectic symmetry are gathered. These formulas are used in Section 4 to study the corresponding global density asymptotic expansions, and in Section 5 to study the soft edge density asymptotic expansions. In Section 6 we use the results of Section 2, 4 and 5 to study our main topics of interest, namely the matching between the bulk and edge asymptotic expansions, and the microscopic origin of the delta functions in (1.4) and its Laguerre analogue.

2 The Gaussian and Laguerre ensembles with unitary symmetry

2.1 Definitions and summary of known results

The Gaussian unitary ensemble consists of the set of Hermitian matrices with diagonal entries distributed according to the normal distribution $\mathrm{N}[0, 1/\sqrt{2}]$ and with upper triangular entries distributed according to $\mathrm{N}[0, 1/2] + i\mathrm{N}[0, 1/2]$. The corresponding eigenvalue p.d.f. is given by

$$\frac{1}{C}\prod_{l=1}^{N} e^{-x_l^2} \prod_{1\le j<k\le N} (x_k - x_j)^2 \tag{2.1}$$

where here and below C denotes *some* normalization constant.

The Laguerre unitary ensemble can be specified by matrices $X = A^\dagger A$ where A is a $n \times m$ $(n \ge m)$ complex Gaussian matrix with entries distributed according to $\mathrm{N}[0, 1/\sqrt{2}]+i\mathrm{N}[0, 1/\sqrt{2}]$. All the eigenvalues are non-negative and have the joint distribution

$$\frac{1}{C}\prod_{l=1}^{m} x_l^{\alpha} e^{-x_l} \prod_{1\le j<k\le m} (x_k - x_j)^2 \tag{2.2}$$

where $\alpha = n - m$.

The eigenvalue p.d.f.s (2.1) and (2.2) are special cases of the functional form

$$\mathrm{UE}_N(g_2) := \frac{1}{C}\prod_{l=1}^{N} g_2(x_l) \prod_{1\le j<k\le N} (x_k - x_j)^2 \tag{2.3}$$

defining a general matrix ensemble with unitary symmetry in terms of its eigenvalue p.d.f. Thus

$$\mathrm{Ev}(\mathrm{GUE}_N) = \mathrm{UE}_N(e^{-x^2}), \qquad \mathrm{Ev}(\mathrm{LUE}_N) = \mathrm{UE}_N(x^{\alpha} e^{-x})$$

where $\mathrm{Ev}(M)$ denotes the eigenvalue p.d.f. of the matrix ensemble M.

It is a basic result in random matrix theory (see e.g. [5]) that the eigenvalue density for the ensemble (2.3) can be written in terms of the monic polynomials $\{p_n(x)\}$ orthogonal with respect to the weight $g_2(x)$. Thus with $(p_n, p_n)_2$ denoting the corresponding normalizations we have

$$\rho(x; \mathrm{UE}_N(g_2)) = g_2(x) \sum_{j=0}^{N-1} \frac{(p_j(x))^2}{(p_j, p_j)_2}. \tag{2.4}$$

In a recent study Kalisch and Braak [10] have obtained the leading correction term to the Wigner semi-circle law for the asymptotic expansion of (2.4) in the case of the GUE.

Proposition 1. *Let $-1 < X < 1$ be fixed. One has*

$$\frac{1}{N}\rho(X; \mathrm{UE}_N(e^{-2Nx^2})) \sim \rho_{\mathrm{W}}(X) - \frac{2\cos(2N\pi P_{\mathrm{W}}(X))}{\pi^3 \rho_{\mathrm{W}}^2(X)}\frac{1}{N} + \mathrm{O}\Big(\frac{1}{N^2}\Big), \tag{2.5}$$

where $\rho_{\mathrm{W}}(x)$ is given by (1.1) and

$$P_{\mathrm{W}}(x) = 1 + \frac{x}{2}\rho_{\mathrm{W}}(x) - \frac{1}{\pi}\mathrm{Arccos}\, x. \tag{2.6}$$

The methods in [10] are particular to the Gaussian ensembles, relying on an integral formula coming from the supersymmetry method. Subsequently the present authors [8] have introduced a different strategy which reclaims (2.5), and furthermore applies equally well to the Laguerre case, for which the following result was obtained.

Proposition 2. *Let $0 < X < 1$ be fixed. We have*

$$\begin{aligned}\frac{1}{N}\rho(X;\mathrm{UE}_N(x^\alpha e^{-4Nx})) &\sim \rho_{\mathrm{MP}}(X)\\ &-\Big(\frac{\cos((2N+\alpha)\pi P_{\mathrm{MP}}(X)-\alpha\pi(1+X\rho_{\mathrm{MP}}(X))}{\pi^3X^2\rho^2_{\mathrm{MP}}(X)}-\frac{\alpha}{\pi^2 X\rho_{\mathrm{MP}}(X)}\Big)\frac{1}{N}+O\Big(\frac{1}{N^2}\Big)\end{aligned} \tag{2.7}$$

where

$$\rho_{\mathrm{MP}}(x) := \frac{2}{\pi}\sqrt{\frac{1}{x}-1}, \qquad P_{\mathrm{MP}}(x) = 1 + x\rho_{\mathrm{MP}}(x) - \frac{2}{\pi}\mathrm{Arccos}\sqrt{x} \tag{2.8}$$

(the subscript MP denotes Marčenko-Pastur, who first derived the limit law giving the leading term in this expression).

The strategy of [8] was to use integral representations of the product of orthogonal polynomials which result from the Christoffel-Darboux summation of (2.4). In addition to yielding the asymptotics of the global bulk density, it also gave asymptotics of the density at the so called soft edge. This is the name given to the boundary of the support at leading order, with the feature that the density, appropriately scaled, is non-zero on both sides of this boundary.

Proposition 3. *Let ξ be fixed. For the GUE*

$$\begin{aligned}\frac{1}{2N^{2/3}}\rho\left(1+\frac{\xi}{2N^{2/3}};\mathrm{UE}_N(e^{-2Nx^2})\right) &= [\mathrm{Ai}'(\xi)]^2 - \xi[\mathrm{Ai}(\xi)]^2\\ &-\frac{1}{20}\left(3\xi^2[\mathrm{Ai}(\xi)]^2 - 2\xi[\mathrm{Ai}'(\xi)]^2 - 3\mathrm{Ai}(\xi)\mathrm{Ai}'(\xi)\right)\frac{1}{N^{2/3}} + O\left(\frac{1}{N}\right),\end{aligned} \tag{2.9}$$

while for the LUE

$$\begin{aligned}&\frac{1}{(2N)^{2/3}}\rho\left(1+\frac{\xi}{(2N)^{2/3}};\mathrm{UE}_N(x^\alpha e^{-4Nx})\right)\\ &= ([\mathrm{Ai}'(\xi)]^2 - \xi[\mathrm{Ai}(\xi)]^2) + \frac{\alpha}{2^{1/3}}[\mathrm{Ai}(\xi)]^2\frac{1}{N^{1/3}}\\ &+\frac{2^{1/3}}{10}\left(3\xi^2[\mathrm{Ai}(\xi)]^2 - 2\xi[\mathrm{Ai}'(\xi)]^2 + (2-5\alpha^2)\mathrm{Ai}(\xi)\mathrm{Ai}'(\xi)\right)\frac{1}{N^{2/3}} + O\left(\frac{1}{N}\right).\end{aligned} \tag{2.10}$$

2.2 Matching of the bulk and edge expansions

We pursue the matching phenomenon observed in [8] between the asymptotic expansion of the bulk density, expanded about the soft edge, and the asymptotic expansion of the edge density, expanded into the bulk. Explicitly, it was found that setting

$$X = 1 + \xi/2N^{2/3}, \qquad (\xi < 0) \tag{2.11}$$

in the asymptotic expansion (2.5) multiplied by $N^{2/3}/2$, and then expanding in N, a matching is obtained with the $\xi \to -\infty$ asymptotic expansion of the right hand side of (2.9). This was checked on terms in ξ in the latter accessible by expanding the former; on this point we note that terms of all orders in inverse powers of N in (2.5), will after substitution of (2.11) contribute to each term in the expansion of (2.9). A similar matching was observed between the bulk LUE density expanded with the substitution (2.11), and the $\xi \to -\infty$ asymptotic expansion of the $O(1)$ and $O(N^{-1/3})$ terms of the edge density (2.10).

Based on this evidence, the hypothesis was put forward in [8] that the matching persists between all terms in ξ, and at all orders in inverse (fractional) powers of N. Here we probe this hypothesis further by extending the asymptotic expansions (2.5), (2.7), (2.9) and (2.10). As explained in [8], the orthogonal polynomial method readily allows for the computation of higher order terms, which we compute to be given as follows.

Proposition 4. *The $O(1/N^2)$ term in (2.5) is*

$$\left(\frac{1}{16\pi(1-X^2)^{5/2}} + \frac{X(15+2X^2)\sin[2N\pi P_{\mathrm{W}}(X)]}{48\pi(1-X^2)^{5/2}}\right)\frac{1}{N^2}; \tag{2.12}$$

the $O(1/N^3)$ term in (2.5) is

$$\frac{180+981X^2+60X^4+4X^6}{2304\pi(1-X^2)^4}\cos[2N\pi P_{\mathrm{W}}(X)]\frac{1}{N^3}; \tag{2.13}$$

the oscillatory $O(1/N^4)$ term in (2.5) is

$$-X\frac{323190+647055X^2+20358X^4+6084X^6-1112X^8}{829440\pi(1-X^2)^{11/2}}\sin[2N\pi P_{\mathrm{W}}(X)]\frac{1}{N^4}; \tag{2.14}$$

the $O(1/N^2)$ term in (2.7) is

$$\begin{aligned}\Bigg(&\left(\frac{\alpha}{8\pi(1-X)^2}\right)\cos[2N\pi P_{\mathrm{MP}}(X) - 2\alpha \operatorname{Arccos}\sqrt{X}]\\ &+\left(\frac{-3+12X+8X^2+12(-1+X)(-1+2X)\alpha^2}{192\pi(1-X)^{5/2}X^{3/2}}\right)\sin[2N\pi P_{\mathrm{MP}}(X) - 2\alpha \operatorname{Arccos}\sqrt{X}]\\ &+\frac{1+4(-1+X)\alpha^2}{64\pi(1-X)^{5/2}X^{3/2}}\Bigg)\frac{1}{N^2};\end{aligned} \tag{2.15}$$

the $O(1/N)$ term in (2.9) is

$$\left(\left(-\frac{\xi}{12}+\frac{3\xi^4}{40}+\frac{\xi^7}{48}\right)(\mathrm{Ai}(\xi))^2 - \frac{3\xi^2}{40}\mathrm{Ai}(\xi)\mathrm{Ai}'(\xi) + \left(\frac{1}{12}-\frac{\xi^3}{20}-\frac{\xi^6}{48}\right)(\mathrm{Ai}'(\xi))^2\right)\frac{1}{N}; \tag{2.16}$$

the $O(1/N)$ term in (2.10) is

$$\alpha\left(\left(-\frac{7\xi}{15}+\frac{\alpha^2\xi}{6}\right)(\mathrm{Ai}(\xi))^2 - \frac{\xi^2}{5}\mathrm{Ai}(\xi)\mathrm{Ai}'(\xi) + \left(-\frac{1}{6}+\frac{\alpha^2}{6}\right)(\mathrm{Ai}'(\xi))^2\right)\frac{1}{N}. \tag{2.17}$$

Considering first the GUE, we now substitute (2.11) in (2.5) extended by (2.12), (2.13), (2.14). Expanding the asymptotic series (an operation we denote by $\dot\sim$) gives the new asymptotic series in N,

$$
\begin{aligned}
&\frac{1}{N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{UE}_N(e^{-2Nx^2})\Big)\\
&\dot\sim\Big(\frac{2\sqrt{|\xi|}}{\pi}+\frac{1}{16\pi|\xi|^{5/2}}+\Big(-\frac{1}{2\pi|\xi|}+\frac{1225}{2304\pi\xi^4}\Big)\cos(4|\xi|^{3/2}/3)-\frac{17\sin(4|\xi|^{3/2}/3)}{48\pi|\xi|^{5/2}}\Big)\\
&\quad+\Big(-\frac{|\xi|^{3/2}}{4\pi}+\frac{5}{128\pi|\xi|^{3/2}}+\Big(-\frac{43}{480\pi}-\frac{23695}{331776\pi|\xi|^3}\Big)\cos(4|\xi|^{3/2}/3)\\
&\quad+\Big(\frac{233}{4608}-\frac{|\xi|^3}{20}\Big)\frac{\sin(4|\xi|^{3/2}/3)}{\pi|\xi|^{3/2}}\Big)\frac{1}{N^{2/3}}\\
&\quad+O\Big(\frac{1}{N^{4/3}}\Big).
\end{aligned}
\tag{2.18}
$$

On the other hand, making use of the asymptotic series [13]

$$
\mathrm{Ai}(-z)\ \sim\ \pi^{-1/2}z^{-1/4}\Big(\sin(\zeta+\pi/4)\sum_{k=0}^{\infty}(-1)^k c_{2k}\zeta^{-2k}-\cos(\zeta+\pi/4)\sum_{k=0}^{\infty}(-1)^k c_{2k+1}\zeta^{-2k-1}\Big),
\tag{2.19}
$$

where

$$
\zeta=\frac{2}{3}z^{3/2},\qquad c_0=1,\qquad c_k=\frac{(2k+1)(2k+3)\cdots(6k-1)}{(216)^k k!}\ (k\ge 1),
$$

the $\xi\to-\infty$ expansion of the $O(1)$ and $O(1/N^{2/3})$ terms in (2.9) can readily be computed. Agreement is found with all terms in (2.18) except the one involving the fraction 23695/331776. Thus, even though no terms $O(1/N)$ have yet appeared, whereas (2.16) has a term at this order, the evidence is still in favor of a matching between all terms in ξ, and at all orders in inverse powers of N. However this matching cannot be fully exhibited at any order in N in the expansion of (2.9) without knowledge of all terms in the asymptotic expansion of (2.5).

As already mentioned, a similar matching phenomenon was observed in [8] in the case of the LUE, and conjectured to hold at general orders as for the GUE. Further evidence for this conjecture can be obtained by substituting (2.11) in (2.7) extended by (2.15), expanding as a series in inverse powers of N, and comparing against the $\xi\to-\infty$ expansion of (2.10). The former operation gives

$$
\begin{aligned}
&\frac{1}{N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{UE}_N(x^{\alpha}e^{-4Nx})\Big)\\
&\quad\dot\sim\Big(\frac{2^{2/3}|\xi|^{1/2}}{\pi}-\frac{\cos(4|\xi|^{3/2}/3)}{2^{4/3}\pi|\xi|}\Big)+\frac{\alpha(1+\sin(4|\xi|^{3/2}/3))}{2^{2/3}\pi|\xi|^{1/2}N^{1/3}}\\
&\quad\ +\frac{1}{160\pi|\xi|^{3/2}N^{2/3}}\Big(5-20\alpha^2+80|\xi|^3+40(-1+2\alpha^2)|\xi|^{3/2}\cos(4|\xi|^{3/2}/3)\\
&\quad\ +(5-20\alpha^2+16|\xi|^3)\sin(4|\xi|^{3/2}/3)\Big)+O\Big(\frac{1}{N}\Big).
\end{aligned}
\tag{2.20}
$$

The latter operation gives agreement with this expansion at $O(1)$, $O(1/N^{1/3})$ and for the terms involving factors of $|\xi|^3$ at $O(1/N^{2/3})$. This is consistent with the matching hypothesis.

3 The Gaussian and Laguerre ensembles with orthogonal and symplectic symmetry – general formulas

The Gaussian orthogonal ensemble has already been defined in the Introduction. At the level of an eigenvalue p.d.f., the GOE can be defined by the joint distribution

$$\frac{1}{C}\prod_{l=1}^{N} e^{-x_l^2/2} \prod_{1\le j<k\le N} |x_k - x_j|.$$

Likewise the Gaussian symplectic ensemble, Laguerre orthogonal ensemble and Laguerre symplectic ensemble can be specified either in terms of the distribution of certain classes of random matrices (real symmetric matrices in the cases of orthogonal symmetry, and Hermitian matrices with real quaternion elements in the cases of symplectic symmetry), or in terms of the functional form of the eigenvalue p.d.f. (see e.g. [5]). Here we will note only the latter, which in the case of the GSE reads

$$\frac{1}{C}\prod_{l=1}^{N} e^{-2x_l^2} \prod_{1\le j<k\le N} (x_k - x_j)^4;$$

for the LOE reads

$$\frac{1}{C}\prod_{l=1}^{N} x_l^{\alpha/2} e^{-x_l/2} \prod_{1\le j<k\le N} |x_k - x_j|;$$

and for the LSE reads

$$\frac{1}{C}\prod_{l=1}^{N} x_l^{2\alpha} e^{-2x_l} \prod_{1\le j<k\le N} (x_k - x_j)^4.$$

For the Laguerre ensembles one requires the eigenvalues be positive and thus $x_l > 0$ $(l = 1,\dots,N)$. Thus we see that if we define a matrix ensemble with orthogonal and symplectic symmetry by the eigenvalue p.d.f.s

$$\mathrm{OE}_N(g_1) := \frac{1}{C}\prod_{l=1}^{N} g_1(x_l) \prod_{1\le j<k\le N} |x_k - x_j|$$

and

$$\mathrm{SE}_N(g_4) := \frac{1}{C}\prod_{l=1}^{N} g_4(x_l) \prod_{1\le j<k\le N} (x_k - x_j)^4$$

respectively, then we have

$$\begin{aligned} &\mathrm{Ev}(\mathrm{GOE}_N) = \mathrm{OE}_N(e^{-x^2/2}), \quad \mathrm{Ev}(\mathrm{LOE}_N) = \mathrm{OE}_N(x^{\alpha/2}e^{-x/2}) \\ &\mathrm{Ev}(\mathrm{GSE}_N) = \mathrm{SE}_N(e^{-2x^2}), \qquad \mathrm{Ev}(\mathrm{LSE}_N) = \mathrm{SE}_N(x^{2\alpha}e^{-x}). \end{aligned} \tag{3.1}$$

In the work [1], the eigenvalue densities for the ensembles $\mathrm{OE}(g_1)$ and $\mathrm{SE}(g_4)$ were computed for all the so called classical weights

$$g_1(x) = \begin{cases} e^{-x^2/2}, & \text{Gaussian} \\ x^{(\alpha-1)/2}e^{-x/2}\ (x>0), & \text{Laguerre} \\ (1-x)^{(a-1)/2}(1+x)^{(b-1)/2}\ (-1<x<1), & \text{Jacobi} \\ (1+x^2)^{-(\alpha+1)/2}, & \text{Cauchy} \end{cases} \tag{3.2}$$

$$g_4(x) = \begin{cases} e^{-x^2}, & \text{Hermite} \\ x^{\alpha+1}e^{-x}, & \text{Laguerre} \\ (1-x)^{a+1}(1+x)^{b+1}, & \text{Jacobi} \\ (1+x^2)^{-(\alpha-1)}, & \text{Cauchy} \end{cases} \tag{3.3}$$

in terms of a formula depending on the symmetry class only. Thus for ensembles $\mathrm{OE}_N(g_1)$ with classical weights (3.2) one has

$$\begin{aligned} \rho(x; \mathrm{OE}_N(g_1(x))) &= \rho(x; \mathrm{UE}_{N-1}(g_2(x))) \\ &+ \frac{c_{N-2}}{(p_{N-2}, p_{N-2})_2 (p_{N-1}, p_{N-1})_2} g_1(x) p_{N-1}(x) \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{sgn}(x-t) p_{N-2}(t) g_1(t)\, dt, \end{aligned} \tag{3.4}$$

while for ensembles $\mathrm{SE}_N(g_4)$ with classical weights (3.3) one has

$$\begin{aligned} \rho(x; \mathrm{SE}_N(g_4(x))) &= \frac{1}{2}\rho(x; \mathrm{UE}_{2N}(g_2)) \\ &\quad - g_1(x) \frac{c_{2N-1} p_{2N}(x)}{2(p_{2N}, p_{2N})_2 (p_{2N-1}, p_{2N-1})_2} \int_x^{\infty} g_1(t) p_{2N-1}(t)\, dt. \end{aligned} \tag{3.5}$$

In (3.4) and (3.5),

$$g_2(x) = \begin{cases} e^{-x^2}, & \text{Hermite} \\ x^{\alpha}e^{-x}, & \text{Laguerre} \\ (1-x)^{a}(1+x)^{b}, & \text{Jacobi} \\ (1+x^2)^{-\alpha}, & \text{Cauchy} \end{cases} \tag{3.6}$$

while

$$\frac{c_j}{(p_j, p_j)_2} = \begin{cases} 1, & \text{Hermite} \\ \frac{1}{2}, & \text{Laguerre.} \end{cases} \tag{3.7}$$

The quantities $\{p_n(x)\}$ are the monic classical orthogonal polynomials with respect to the weights (3.6), and $(p_n, p_n)_2$ the corresponding normalizations. Thus in the Gaussian case

$$p_n(x) = 2^{-n} H_n(x), \qquad (p_n, p_n)_2 = \pi^{1/2} 2^{-n} n! \tag{3.8}$$

while in the Laguerre case

$$p_n(x) = (-1)^n n! L_n^{\alpha}(x), \qquad (p_n, p_n)_2 = \Gamma(n+1)\Gamma(\alpha+n+1). \tag{3.9}$$

Essential tools in our subsequent analysis of the asymptotic forms of (3.4) and (3.5) are particular asymptotic formulas for the Hermite and Laguerre polynomials. Consider first the bulk region. In the case of the Hermite polynomials, the formula is due to Plancherel and Rotach [14]. It tells us that with

$$x = (2n+1)^{1/2} \cos\phi, \qquad \epsilon \le \phi \le \pi - \epsilon,$$

we have

$$e^{-x^2/2} H_n(x) = 2^{n/2+1/4} (n!)^{1/2} (\pi n)^{-1/4} (\sin\phi)^{-1/2} \Big(\sin\big((n/2+1/4)(\sin 2\phi - 2\phi) + 3\pi/4\big) + O(n^{-1}) \Big).$$

Setting

$$\sqrt{2N}X = (2(N+m)+1)^{1/2}\cos\phi$$

with $-1 < X < 1$ fixed we deduce from this that for m fixed

$$H_{N+m}(\sqrt{2N}X) = \Big(\frac{2}{\pi}\Big)^{1/4}\frac{2^{m/2+N/2}}{(1-X^2)^{1/4}}N^{m/2-1/4}(N!)^{1/2}e^{NX^2}g_{m,N}^{(H)}(X)\Big(1+O\Big(\frac{1}{N}\Big)\Big) \tag{3.10}$$

where

$$g_{m,N}^{(H)}(x) := \cos\Big(Nx\sqrt{1-x^2}+(N+1/2)\mathrm{Arcsin}\,x - N\pi/2 - m\mathrm{Arccos}\,x\Big). \tag{3.11}$$

The Plancherel-Rotach formula (3.10) was extended by Moecklin to the Laguerre polynomials [11]. With

$$x = (4n+2\alpha+2)\cos^2\phi, \qquad \epsilon \le \phi \le \pi/2 - \epsilon n^{-1/2}$$

it reads

$$\begin{aligned} e^{-x/2}L_n^{(\alpha)}(x) \quad &= \quad (-1)^n(\pi\sin\phi)^{-1/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4} \\ &\quad \times\Big(\sin\Big((n+(\alpha+1)/2)(\sin 2\phi - 2\phi)+3\pi/4\Big)+(nx)^{-1/2}O(1)\Big). \end{aligned}$$

Setting

$$4nX = (4(n+m)+2\alpha+2)\cos^2\phi$$

with m fixed and $\epsilon/n < X < 1$ we deduce from this that

$$x^{\alpha/2}e^{-x/2}L_{n+m}^{(\alpha)}(x)\Big|_{x=4nX} = (-1)^{n+m}(2\pi\sqrt{X(1-X)})^{-1/2}n^{\alpha/2-1/2}\Big(g_{m,n}^{(L)}(X)+O(\frac{1}{n})\Big) \tag{3.12}$$

where

$$g_{m,n}^{(L)}(X) := \sin\Big(2n(\sqrt{X(1-X)}-\mathrm{Arccos}\sqrt{X})-(2m+\alpha+1)\mathrm{Arccos}\sqrt{X}+3\pi/4\Big). \tag{3.13}$$

We turn our attention now to the (soft) spectrum edge. In the Hermite case, the formula of Plancherel and Rotach tells us that with

$$x = (2N)^{1/2}+2^{-1/2}N^{-1/6}t \tag{3.14}$$

we have

$$\exp(-x^2/2)H_N(x) = \pi^{1/4}2^{N/2+1/4}(N!)^{1/2}N^{-1/12}\Big\{\mathrm{Ai}(t)+O(N^{-2/3})\Big\},$$

where $\mathrm{Ai}(t)$ denotes the Airy function. It follows from this that with x again related to t by (3.14)

$$\exp(-x^2/2)H_{N+m}(x) = (2N)^{m/2}\pi^{1/4}2^{N/2+1/4}(N!)^{1/2}N^{-1/12}\Big\{\mathrm{Ai}(t)-\frac{m}{N^{1/3}}\mathrm{Ai}'(t)+O(N^{-2/3})\Big\}. \tag{3.15}$$

In the Laguerre case, Szegö [16] gives that with

$$x = 4N+2\alpha+2+2(2N)^{1/3}t \tag{3.16}$$

we have

$$e^{-x/2}L_N^{\alpha}(x) = (-1)^N 2^{-\alpha-1/3}N^{-1/3}\Big(\mathrm{Ai}(t) + O(N^{-2/3})\Big).$$

It then follows that for fixed p

$$e^{-x/2}L_{N+p}^{\alpha}(x)\Big|_{x=4N+2(2N)^{1/3}\xi} = (-1)^{N+p}2^{-\alpha-1/3}N^{-1/3}\Big(\mathrm{Ai}(\xi) - \frac{2p+\alpha+1}{(2N)^{1/3}}\mathrm{Ai}'(\xi) + O(N^{-2/3})\Big). \tag{3.17}$$

4 Asymptotic expansions in the bulk

4.1 GOE and GSE in the bulk

As with the GUE, the asymptotic expansion of the bulk density for the GOE and GSE has been carried out by Kalisch and Braak [10]. But as their method is particular to the Gaussian case, we give an alternative method which can be extended to the Laguerre case.

Consider first the GOE. Substituting the appropriate formula from (3.7), and (3.8), in (3.4) we see after minor manipulation that

$$\rho(x;\mathrm{OE}_N(e^{-x^2/2})) = \rho(x;\mathrm{UE}_{N-1}(e^{-x^2})) + \frac{e^{-x^2/2}H_{N-1}(x)}{2^{N-1}\pi^{1/2}(N-2)!}\int_0^x e^{-t^2/2}H_{N-2}(t)\,dt. \tag{4.1}$$

Also, making use of (2.4) shows

$$\rho(x;\mathrm{UE}_{N-1}(e^{-x^2})) = \rho(x;\mathrm{UE}_N(e^{-x^2})) - \frac{2^{-(N-1)}e^{-x^2}}{\pi^{1/2}(N-1)!}(H_{N-1}(x))^2. \tag{4.2}$$

From (4.1) and (4.2) the following asymptotic formula for the bulk eigenvalue density is obtained.

Proposition 5. *Let $-1 < X < 1$ be fixed. We have*

$$\frac{1}{N}\rho(X;\mathrm{OE}_N(e^{-Nx^2})) \ \sim\ \frac{2}{\pi}\sqrt{1-X^2} - \frac{1}{2\pi N\sqrt{1-X^2}} + O\Big(\frac{1}{N^2}\Big). \tag{4.3}$$

Proof. First we note that

$$\rho(X;\mathrm{OE}_N(e^{-Nx^2})) = \sqrt{2N}\rho(\sqrt{2N}X;\mathrm{OE}_N(e^{-x^2/2})), \tag{4.4}$$

and we proceed to analyze the large N form of the right hand side using (4.1). In relation to the latter, by a simple change of variables,

$$\int_0^{\sqrt{2N}X} e^{-t^2/2}H_{N-2}(t)\,dt = \sqrt{2N}\int_0^X e^{-t^2/2}H_{N-2}(t)\Big|_{t=\sqrt{2N}T}dT,$$

while making use of (3.10) shows

$$\begin{aligned}&\sqrt{2N}\int_0^X e^{-t^2/2}H_{N-2}(t)\Big|_{t=\sqrt{2N}T}dT\\ &\quad\sim \sqrt{2N}\Big(\frac{2}{\pi}\Big)^{1/4}\frac{2^{-1+N/2}}{(1-T^2)^{1/4}}N^{-5/4}(N!)^{1/2}\int_0^X \frac{g_{-2,N}^{(H)}(T)}{(1-T^2)^{1/4}}\,dT.\end{aligned} \tag{4.5}$$

The leading contribution to the integral for large N comes from the neighborhood of the endpoints $T = 0$ and $T = X$. About $T = 0$

$$g^{(H)}_{m,N}(T) \ \sim \ \cos\Big(2NT - (N+m)\pi/2 + O(T^2)\Big),$$

while about $T = X$

$$\begin{aligned} g^{(H)}_{m,N}(T) \quad &\sim \quad \cos\Big(NX\sqrt{1-X^2} + (N+1/2)\mathrm{Arcsin}\, X \\ &\qquad -N\pi/2 - m\mathrm{Arccos}\, X + 2N\sqrt{1-X^2}(T-X) + O((T-X)^2)\Big). \end{aligned}$$

Thus we have

$$\int_0^X \frac{g^{(H)}_{-2,N}(T)}{(1-T^2)^{1/4}}\, dT \ \sim \ \frac{1}{2N(1-X^2)^{3/4}} \tilde{g}^{(H)}_{-2,N}(X) \tag{4.6}$$

where

$$\tilde{g}^{(H)}_{m,N}(x) := \sin\Big(Nx\sqrt{1-x^2} + (N+1/2)\mathrm{Arcsin}\, x - N\pi/2 - m\mathrm{Arccos}\, x\Big) \tag{4.7}$$

and use has been made of the fact that N is assumed even in (3.4).

We read off from (3.10) that

$$e^{-x^2}H_{N-1}(x)\Big|_{x=\sqrt{2N}X} = \Big(\frac{2}{\pi}\Big)^{1/4} \frac{2^{-1/2+N/2}}{(1-X^2)^{1/4}} N^{-3/4}(N!)^{1/2} g^{(H)}_{-1,N}(X)\Big(1 + O\Big(\frac{1}{N}\Big)\Big). \tag{4.8}$$

Making use of this together with (4.5), (4.6) and Stirling's formula we deduce

$$\begin{aligned} &\frac{e^{-x^2/2}H_{N-1}(x)}{2^{N-1}\pi^{1/2}(N-2)!} \int_0^{\sqrt{2N}X} e^{-t^2/2} H_{N-2}(t)\, dt \\ &\quad \sim \frac{1}{\pi\sqrt{2N}} \frac{g^{(H)}_{-1,N}(X)\tilde{g}^{(H)}_{-2,N}(X)}{(1-X^2)}\Big(1 + O\Big(\frac{1}{N}\Big)\Big) \\ &\quad = \frac{1}{\pi\sqrt{2N}} \frac{g^{(H)}_{-1,N}(X)}{(1-X^2)}\Big(X\tilde{g}^{(H)}_{-1,N}(X) + \sqrt{1-X^2}\, g^{(H)}_{-1,N}(X)\Big)\Big(1 + O\Big(\frac{1}{N}\Big)\Big) \end{aligned} \tag{4.9}$$

where the equality follows from simple trigonometric identities.

For the second term in (4.2), use of (4.8) shows

$$-\frac{2^{-(N-1)}e^{-x^2}}{\pi^{1/2}(N-1)!}(H_{N-1}(x))^2\Big|_{x=\sqrt{2N}X} \ \sim \ -\frac{1}{\pi}\sqrt{\frac{2}{N}} \frac{(g^{(H)}_{-1,N}(X))^2}{\sqrt{1-X^2}}\Big(1 + O\Big(\frac{1}{N}\Big)\Big). \tag{4.10}$$

And for the first term in (4.2) we know from (2.5) that

$$\begin{aligned} &\rho(\sqrt{2N}X; \mathrm{UE}_N(e^{-x^2})) \\ &\quad \sim \frac{\sqrt{2N}}{\pi}\sqrt{1-X^2} - \sqrt{\frac{2}{N}} \frac{\cos(2NX\sqrt{1-X^2} + 2N\mathrm{Arcsin}\, X - N\pi)}{2\pi(1-X^2)} + O\Big(\frac{1}{N^{3/2}}\Big). \end{aligned} \tag{4.11}$$

But

$$\cos(2NX\sqrt{1-X^2} + 2N\mathrm{Arcsin}\, X - N\pi) = \sqrt{1-X^2}(2(\tilde{g}^{(H)}_{-1,N}(X))^2 - 1) - 2X\tilde{g}^{(H)}_{-1,N}(X) g^{(H)}_{-1,N}(X). \tag{4.12}$$

Substituting (4.12) in (4.11), then adding this, (4.9), and (4.10), and recalling (4.4) gives (4.3). □

The result (4.3) agrees with that computed by Kalisch and Braak in [10] and is also consistent with (1.3). We remark that in [10] the $O(1/N^2)$ term is also given, being equal to

$$\frac{3+4X^2}{16\pi(1-X^2)^{5/2}N^2} - \frac{\cos((2N-1)\mathrm{Arcsin}\, X + 2NX\sqrt{1-X^2})}{8\pi(1-X^2)^{5/2}N^2}. \tag{4.13}$$

In principle the present method offers a systematic approach to all correction terms. For this we need the explicit form of the higher order terms in (3.10), and these can in fact be calculated from the results in [14]. However we have not pursued such calculations. We remark too that a calculation of the non-oscillatory portion of (4.13) is undertaken in [2]; however the result obtained does not agree with (4.13).

We turn our attention now to the GSE. First we note that

$$\rho(X; \mathrm{SE}_N(e^{-4Nx^2})) = 2\sqrt{N}\rho(2\sqrt{N}X; \mathrm{SE}_N(e^{-x^2})). \tag{4.14}$$

Regarding the right hand side, making use of (3.7), (3.8) as well as the integral evaluation (see e.g. [1])

$$2^{-N}\int_0^\infty e^{-t^2/2}H_N(t)\,dt = \sqrt{\frac{\pi}{2}}\frac{N!}{2^N(N/2)!} \tag{4.15}$$

gives

$$\begin{aligned}\rho(x; \mathrm{SE}_N(e^{-x^2})) &= \frac{1}{2}\rho(x; \mathrm{UE}_{2N}(e^{-x^2})) \\ &\quad - \frac{e^{-x^2/2}H_{2N}(x)}{4\pi^{1/2}(2N-1)!}\Big(\sqrt{\frac{\pi}{2}}\frac{(2N-1)!}{2^{2N-1}(N-1/2)!} - 2^{-(2N-1)}\int_0^x e^{-t^2/2}H_{2N-1}(t)\,dt\Big).\end{aligned} \tag{4.16}$$

The asymptotic form of (4.16) can be calculated according to the strategy of the proof of Proposition 5 to give the following result for the bulk scaled density in the GSE.

Proposition 6. *Let $-1 < X < 1$ be fixed, and $g_{0,2N}^{(H)}(X)$ be given according to (3.11). We have*

$$\begin{aligned}\frac{1}{N}\rho(X; \mathrm{SE}_N(e^{-4Nx^2})) \;\sim\; & \frac{2}{\pi}\sqrt{1-X^2} - \Big(\frac{1}{\sqrt{2\pi N}} + \frac{(-1)^N}{2\pi N}\Big)\frac{g_{0,2N}^{(H)}(X)}{(1-X^2)^{1/4}} \\ & + \frac{1}{4\pi N}\frac{1}{\sqrt{1-X^2}} + O\Big(\frac{1}{N^{3/2}}\Big).\end{aligned} \tag{4.17}$$

Proof. Analogous to (4.5), it follows from (3.10) that

$$\int_0^{2\sqrt{N}X} e^{-t^2/2}H_{2N-1}(t)\,dt \;\sim\; 2\sqrt{N}\Big(\frac{2^{N-1}}{\pi^{1/4}N^{3/4}}((2N)!)^{1/2}\Big)\int_0^X \frac{g_{-1,2N}^{(H)}(t)}{(1-t^2)^{1/4}}\,dt,$$

while proceeding as in the derivation of (4.6) shows

$$\int_0^X \frac{g_{-1,2N}^{(H)}(t)}{(1-t^2)^{1/4}}\,dt \;\sim\; \frac{1}{4N}\frac{1}{(1-X^2)^{3/4}}\tilde{g}_{-1,2N}^{(H)}(X) - \frac{(-1)^N}{4N}.$$

Thus, after making use too of Stirling's formula,

$$\Big(\sqrt{\frac{\pi}{2}}\frac{(2N-1)!}{2^{2N-1}(N-1/2)!} - 2^{-(2N-1)}\int_0^{\sqrt{2N}X} e^{-t^2/2}H_{2N-1}(t)\,dt\Big)$$
$$\sim (N-1/2)!\Big(\frac{1}{\sqrt{2N}} - \frac{2}{\sqrt{\pi}}\Big(\frac{1}{4N}\frac{1}{(1-X^2)^{3/4}}\tilde{g}^{(H)}_{-1,2N}(X) - \frac{(-1)^N}{4N}\Big)\Big).$$

Since (3.10) gives

$$e^{-x^2/2}H_{2N}(x)\Big|_{x=2\sqrt{N}X} = \pi^{-1/4}\frac{2^N N^{-1/4}}{(1-X^2)^{1/4}}((2N)!)^{1/2}\Big(g^{(H)}_{0,2N}(X) + O\Big(\frac{1}{N}\Big)\Big)$$

we thus have that with $x = 2\sqrt{N}X$ the final line in (4.16) has the asymptotic behavior

$$-\frac{1}{4\sqrt{\pi}}\frac{g^{(H)}_{0,2N}(X)}{(1-X^2)^{1/4}}\Big(\sqrt{2} - \frac{1}{\sqrt{\pi N}}\Big(\frac{1}{(1-X^2)^{3/4}}\Big(X\tilde{g}^{(H)}_{0,2N}(X) + \sqrt{1-X^2}g^{(H)}_{0,2N}(X)\Big) - (-1)^N\Big).$$

Further, we see from (2.5) and (3.11) that

$$\frac{1}{2}\rho(2\sqrt{N}X; \mathrm{UE}_{2N}(e^{-x^2}))$$
$$\sim \frac{\sqrt{N}}{\pi}\sqrt{1-X^2} - \frac{\sqrt{1-X^2}(2(g^{(H)}_{0,2N}(X))^2 - 1) + 2X\tilde{g}^{(H)}_{0,2N}(X)g^{(H)}_{0,2N}(X)}{8\pi\sqrt{N}(1-X^2)} + O\Big(\frac{1}{N^{3/2}}\Big).$$

Adding these last two results, and recalling (4.16) and (4.14) gives the stated formula. □

In [10], at $O(1/N)$ only the non-oscillatory term is reported. We note too that the non-oscillatory term at $O(1/N)$ in (4.17) is consistent with (1.4) in the case $\beta = 4$.

4.2 The LOE and LSE in the bulk

We now apply the same strategy to the Laguerre case. For the LOE, substituting the appropriate formula from (3.7), and (3.9), into (3.4) shows

$$\rho(x; \mathrm{OE}_N(x^{(\alpha-1)/2}e^{-x/2})) = \rho(x; \mathrm{UE}_{N-1}(x^\alpha e^{-x})) + \frac{(N-1)!}{4(\alpha+N-2)!}x^{(\alpha-1)/2}e^{-x/2}L^\alpha_{N-1}(x)$$
$$\times\Big(\int_0^\infty L^\alpha_{N-2}(t)t^{(\alpha-1)/2}e^{-t/2}\,dt - 2\int_0^x L^\alpha_{N-2}(t)t^{(\alpha-1)/2}e^{-t/2}\,dt\Big) \tag{4.18}$$

while

$$\rho(x; \mathrm{UE}_{N-1}(x^\alpha e^{-x})) = \rho(x; \mathrm{UE}_N(x^\alpha e^{-x})) - \frac{(N-1)!}{\Gamma(N+\alpha)}\Big(x^{\alpha/2}e^{-x/2}L^\alpha_{N-1}(x)\Big)^2. \tag{4.19}$$

Proposition 7. *Let $0 < X < 1$. We have*

$$\frac{1}{N}\rho(X; \mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})) \sim \rho_{\mathrm{MP}}(X) - \frac{1}{2\pi N}\frac{1-\alpha}{\sqrt{X(1-X)}} + o(N^{-1}). \tag{4.20}$$

Proof. By a change of variables

$$\rho(X; \mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})) = 4N\rho(4NX; \mathrm{OE}_N(x^{(\alpha-1)/2}e^{-x/2})), \tag{4.21}$$

so the task is to analyze the $N \to \infty$ asymptotics of the right hand side of (4.18) with x replaced by $4NX$.

We know from [12, 1] that

$$\int_0^\infty L_{N-2}^a(t)t^{(a-1)/2}e^{-t/2}\,dt = \frac{\Gamma((N+1)/2)\Gamma(a+N-1)}{2^{a/2-3/2}\Gamma(N)\Gamma((a+N)/2)} \sim 2N^{(a-1)/2}. \tag{4.22}$$

Regarding the second integral in (4.18), we first write

$$\int_0^{4NX} L_{N-2}^a(t)t^{(a-1)/2}e^{-t/2}\,dt = \Big(\int_0^{4\epsilon} + \int_{4\epsilon}^{4NX}\Big)L_{N-2}^a(t)t^{(a-1)/2}e^{-t/2}\,dt$$

where $0 < \epsilon \ll 1$. In relation to the region $[4\epsilon, 4NX]$, (3.12) tells us that for N even

$$L_{N-2}^\alpha(t)t^{a/2}e^{-t/2}\Big|_{t\mapsto 4NT} = (2\pi\sqrt{T(1-T)})^{-1/2}N^{(\alpha-1)/2}\Big(g_{-2,N}^{(L)}(T) + O\Big(\frac{1}{N}\Big)\Big).$$

Substituting in the integral and changing variables $T = s^2$ shows

$$\int_{4\epsilon}^{4NX} L_{N-2}^\alpha(t)t^{(\alpha-1)/2}e^{-t/2}\,dt \sim (4N)^{1/2}N^{(\alpha-1)/2}\Big(\frac{2}{\pi}\Big)^{1/2}\int_{\sqrt{\epsilon/N}}^{\sqrt{X}} \frac{1}{\sqrt{s}(1-s^2)^{1/4}}g_{-2,N}^{(L)}(s^2)\,ds. \tag{4.23}$$

In relation to the interval $t \in [0, 4\epsilon]$, we know that for $N \to \infty$ [16]

$$L_{N-2}^\alpha(t)t^{\alpha/2}e^{-t/2} \sim N^{\alpha/2}J_\alpha(2(Nt)^{1/2}),$$

where $J_n(z)$ denotes the Bessel function, and thus

$$\int_0^{4\epsilon} L_{N-2}^a(t)t^{(a-1)/2}e^{-t/2}\,dt \sim N^{a/2}\int_0^{4\epsilon} J_\alpha(2(Nt)^{1/2})\,\frac{dt}{\sqrt{t}}. \tag{4.24}$$

We expect the leading contributions to come from the neighborhood of the upper terminal $s = \sqrt{X}$ in (4.23), and the lower terminal $t = 0$ in (4.24) (the integrands should connect smoothly from the lower terminal of (4.23) to the upper terminal of (4.24)). Since about $s = \sqrt{X}$

$$g_{-2,N}^{(L)}(s^2) \sim \sin\Big(2N(\sqrt{X(1-X)} - \mathrm{Arcos}\sqrt{X}) - (\alpha-3)\mathrm{Arcos}\sqrt{X} + 3\pi/4 + 4N\sqrt{1-X}(s-\sqrt{X})\Big)$$

we have

$$\int^{\sqrt{X}} \frac{g_{-2,N}^{(L)}(s^2)}{\sqrt{s}(1-s^2)^{1/4}}\,ds \sim -\frac{1}{4N\sqrt{1-X}}\frac{\tilde{g}_{-2,N}^{(L)}(X)}{X^{1/4}(1-X)^{1/4}}$$

where

$$\tilde{g}_{m,N}^{(L)}(x) := \cos\Big(2N(\sqrt{x(1-x)} - \mathrm{Arcos}\sqrt{x}) - (\alpha+1+2m)\mathrm{Arcos}\sqrt{x} + 3\pi/4\Big).$$

Also

$$\int_0^\infty J_\alpha(t)\,dt = 1,$$

so we have

$$\int_0 J_\alpha(2(Nt)^{1/2})\,\frac{dt}{\sqrt{t}} \ \sim\ N^{-1/2}.$$

Reading off the asymptotic form of the factor $x^{a/2}e^{-x/2}L^a_{N-1}(x)|_{x=4NX}$ in (4.18) from (3.12) we deduce from this that

$$\rho(4NX;\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-x/2})) \ \sim\ \rho(4NX;\mathrm{UE}_{N-1}(x^\alpha e^{-x})) - \frac{g^{(L)}_{-1,N}(X)\tilde{g}^{(L)}_{-2,N}(X)}{8\pi NX(1-X)}$$

It remains to determine the asymptotic form of (4.19). For this we use the analogue of (4.21), and (3.12), to obtain

$$\rho(4NX;\mathrm{UE}_{N-1}(x^\alpha e^{-x})) \ \sim\ \frac{1}{4N}\rho(X;\mathrm{UE}_N(x^\alpha e^{-2Nx})) - \frac{1}{2\pi}\sqrt{\frac{1}{X(1-X)}}(g_{-1,N}(X))^2.$$

Noting from the definitions and by a simple trigonometric identity that

$$\begin{aligned}\tilde{g}^{(L)}_{-2,N}(X) &= (2X-1)\tilde{g}^{(L)}_{-1,N}(X) - 2\sqrt{(1-X)X}\,g^{(L)}_{-1,N}(X)\\ \tilde{g}^{(L)}_{0,N}(X) &= (2X-1)\tilde{g}^{(L)}_{-1,N}(X) + 2\sqrt{(1-X)X}\,g^{(L)}_{-1,N}(X)\end{aligned}$$

we therefore have

$$\rho(4NX;\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-x/2})) \ \sim\ \frac{1}{4N}\rho(X;\mathrm{UE}_N(x^\alpha e^{-4Nx})) - \frac{1}{8\pi N}\frac{g^{(L)}_{-1,N}(X)\tilde{g}^{(L)}_{0,N}(X)}{X(1-X)}. \qquad (4.25)$$

Now, with

$$A_{N,\alpha}(X) := 2N(\sqrt{X(1-X)} - \mathrm{Arccos}\,\sqrt{X}) - \alpha\mathrm{Arccos}\,\sqrt{X} \qquad (4.26)$$

we have that

$$\begin{aligned}g^{(L)}_{-1,N}(X) &= \sin(A_{N,\alpha}(X) + \mathrm{Arccos}\,\sqrt{X} + 3\pi/4)\\ \tilde{g}^{(L)}_{0,N}(X) &= \cos(A_{N,\alpha}(X) - \mathrm{Arccos}\,\sqrt{X} + 3\pi/4)\end{aligned}$$

and thus

$$g^{(L)}_{-1,N}(X)\tilde{g}^{(L)}_{0,N}(X) = \frac{1}{2}\Big(-\cos(2A_{N,\alpha}(X)) + 2\sqrt{X(1-X)}\Big). \qquad (4.27)$$

Substituting (4.27) in (4.25) and noting from (2.7) that

$$\frac{1}{4N}\rho(X;\mathrm{UE}_N(x^\alpha e^{-4Nx})) \ \sim\ \frac{1}{4}\rho_{\mathrm{MP}}(X) - \frac{\cos 2A_{N,\alpha}(X)}{16\pi X(1-X)N} + \frac{\alpha}{8\pi\sqrt{X(1-X)}N} \qquad (4.28)$$

we obtain (4.20). □

Consider now the LSE. Analogous to (4.21), by a change of variables

$$\rho(X;\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})) = 8N\rho(8NX;\mathrm{SE}_N(x^{\alpha+1}e^{-x})),$$

while (3.5) together with the fact [12, 1]

$$\int_0^\infty e^{-t/2}t^{(\alpha-1)/2}L^\alpha_{2N-1}(t)\,dt = 0 \qquad (4.29)$$

shows that

$$\rho(x;\mathrm{SE}_N(x^{\alpha+1}e^{-x})) = \frac{1}{2}\rho(x;\mathrm{UE}_{2N}(x^{\alpha}e^{-x})) - \frac{\Gamma(1+2N)}{4\Gamma(\alpha+2N)}e^{-x/2}x^{(\alpha-1)/2}L_{2N}^{\alpha}(x)\int_0^x e^{-t/2}t^{(\alpha-1)/2}L_{2N-1}^{\alpha}(t)\,dt. \tag{4.30}$$

Proposition 8. *Let $0<X<1$. In terms of the notation (3.13) and (4.26) we have*

$$\frac{1}{N}\rho(X;\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})) \sim \rho_{\mathrm{MP}}(X) - \frac{1}{2(\pi N)^{1/2}}\frac{g_{0,2N}^{(L)}(X)}{X^{3/4}(1-X)^{1/4}} + \frac{\alpha+1}{4\pi N\sqrt{X(1-X)}} + o(N^{-1}). \tag{4.31}$$

Proof. Following the strategy of the proof of Proposition 7 we find

$$\int_0^{8NX} e^{-t/2}t^{(\alpha-1)/2}L_{2N-1}^{\alpha}(t)\,dt \sim (2N)^{(\alpha-1)/2} + \frac{(2N)^{\alpha/2}}{(2\pi)^{1/2}}\frac{\tilde{g}_{-1,2N}^{(L)}(X)}{2NX^{1/4}(1-X)^{3/4}}.$$

Also, (3.12) and Stirling's formula show

$$\frac{\Gamma(1+2N)}{4\Gamma(\alpha+2N)}e^{-x/2}x^{(\alpha-1)/2}L_{2N}^{\alpha}(x)\Big|_{x=8NX} \sim \frac{1}{16(\pi N)^{1/2}(2N)^{(\alpha-1)/2}}\frac{g_{0,2N}^{(L)}(X)}{X^{3/4}(1-X)^{1/4}}.$$

Substituting in (4.30) gives

$$\rho(8NX;\mathrm{SE}_N(x^{\alpha+1}e^{-x})) \sim \frac{1}{16N}\rho(X;\mathrm{UE}_{2N}(x^{\alpha}e^{-8Nx})) - \frac{1}{16(\pi N)^{1/2}}\frac{g_{0,2N}^{(L)}(X)}{X^{3/4}(1-X)^{1/4}} - \frac{g_{0,2N}^{(L)}(X)\tilde{g}_{-1,2N}^{(L)}(X)}{32\pi NX(1-X)}. \tag{4.32}$$

Analogous to (4.27) we have

$$g_{0,2N}^{(L)}(X)\tilde{g}_{-1,2N}^{(L)}(X) = -\frac{1}{2}\Big(\cos 2A_{2N,\alpha}(X) + 2\sqrt{X(1-X)}\Big).$$

Substituting this together with (4.28) in (4.32) gives (4.31). □

5 Asymptotic expansions at the edge

5.1 The GOE and GSE

The scaled densities $\rho(X;\mathrm{OE}_N(e^{-NX^2}))$ and $\rho(X;\mathrm{SE}_N(e^{-4NX^2}))$ have to leading order their support on $(-1,1)$. We know from previous studies [6, 7] that setting X as specified by (2.11) (with the restriction $\xi<0$ removed) and multiplying by $N^{1/3}$, the limit $N\to\infty$ exists and can be computed explicitly. Here we are interested in computing the first correction, as in the soft edge formula (2.9) for the GUE.

In the case of the GOE, we see from (4.4), (4.1) and (4.2) that in addition to the knowledge of (2.9), an asymptotic formula for $\rho(1+\xi/2N^{2/3};\mathrm{OE}_N(e^{-Nx^2}))$ can be obtained by making use of (3.15).

Proposition 9. *We have*

$$\begin{aligned}&\frac{1}{2N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{OE}_N(e^{-Nx^2})\Big)\\&\quad=(\mathrm{Ai}'(\xi))^2-\xi(\mathrm{Ai}(\xi))^2+\frac{1}{2}\mathrm{Ai}(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(t)\,dt\Big)\\&\qquad+\frac{1}{2N^{1/3}}\mathrm{Ai}'(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(t)\,dt\Big)+O\Big(\frac{1}{N^{2/3}}\Big).\end{aligned}\tag{5.1}$$

Proof. Consider the integral in (4.1). We know from [1] that

$$\int_0^x e^{-t^2/2}H_N(t)\,dt=\sqrt{\frac{\pi}{2}}\frac{N!}{(N/2)!}-\int_x^\infty e^{-t^2/2}H_N(t)\,dt.$$

Replacing N by $N-2$, setting $x=(2N)^{1/2}+2^{-1/2}N^{-1/6}\xi$, making use of (3.15) and simplifying shows that

$$\begin{aligned}&\int_0^{(2N)^{1/2}+2^{-1/2}N^{-1/6}\xi} e^{-t^2/2}H_{N-2}(t)\,dt\\&\quad=\sqrt{\frac{\pi}{2}}\frac{(N-2)!}{(N/2-1)!}\Big(1-\int_\xi^\infty \mathrm{Ai}(y)\,dy+\frac{2}{N^{1/3}}\mathrm{Ai}(\xi)+O\Big(\frac{1}{N^{2/3}}\Big)\Big).\end{aligned}\tag{5.2}$$

Now using (3.15) with $m=-1$, and multiplying with the result (5.2) as required by (4.1) we obtain

$$\begin{aligned}&\Big(\frac{e^{-x^2/2}H_{N-1}(x)}{2^{N-1}\pi^{1/2}(N-2)!}\int_0^x e^{-t^2/2}H_{N-2}(t)\,dt\Big)\Big|_{x=(2N)^{1/2}+2^{-1/2}N^{-1/6}\xi}\\&\quad\sim\frac{N^{1/6}}{2^{1/2}}\Big(\mathrm{Ai}(\xi)+\frac{1}{N^{1/3}}\mathrm{Ai}'(\xi)\Big)\Big(1-\int_\xi^\infty \mathrm{Ai}(y)\,dy+\frac{2}{N^{1/3}}\mathrm{Ai}(\xi)\Big).\end{aligned}$$

According to (4.2), we also require the asymptotic formula

$$-\frac{2^{-N+1}}{\pi^{1/2}(N-1)!}(e^{-x^2/2}H_{N-1}(x))^2\Big|_{x=(2N)^{1/2}+2^{-1/2}N^{-1/6}\xi}\sim-\frac{\sqrt{2}}{N^{1/6}}(\mathrm{Ai}'(\xi))^2,$$

which follows from (3.15). Further

$$\rho((2N)^{1/2}+2^{-1/2}N^{-1/6}\xi;\mathrm{UE}_N(e^{-x^2}))=\frac{1}{(2N)^{1/2}}\rho(1+\xi/2N^{2/3};\mathrm{UE}_N(e^{-2Nx^2}))$$

(cf. (4.4)) so the asymptotic form of the first term on the the right hand side of (4.2) can be read off from (2.9). Doing this, and recalling (4.4), we deduce (5.1). □

We turn our attention now to the GSE. We will analyze (4.16) rewritten to read

$$\rho(x;\mathrm{SE}_N(e^{-x^2}))=\frac{1}{2}\rho(x;\mathrm{UE}_{2N}(e^{-x^2}))-\frac{e^{-x^2/2}H_{2N}(x)}{2^{2N+1}\pi^{1/2}(2N-1)!}\int_x^\infty e^{-t^2/2}H_{2N-1}(t)\,dt.\tag{5.3}$$

Proposition 10. *We have*

$$\begin{aligned}&\frac{1}{(2N)^{2/3}}\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{SE}_N(e^{-4Nx^2})\Big)\\&\quad\sim(\mathrm{Ai}'(\xi))^2-\xi(\mathrm{Ai}(\xi))^2-\frac{1}{2}\mathrm{Ai}(\xi)\Big(\int_\xi^\infty \mathrm{Ai}(t)\,dt-\frac{1}{(2N)^{1/3}}\mathrm{Ai}(\xi)+O(N^{-2/3})\Big).\end{aligned}\tag{5.4}$$

Proof. By a simple change of variables

$$\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{SE}_N(e^{-4Nx^2})\Big)=2\sqrt{N}\rho\Big((4N)^{1/2}+2^{-1/2}(2N)^{-1/6}\xi;\mathrm{SE}_N(e^{-x^2})\Big),\tag{5.5}$$

so we seek the large N asymptotic form of (3.15) with $x\mapsto(4N)^{1/2}+2^{-1/2}(2N)^{-1/6}\xi$. Making use of (3.15) shows

$$\begin{aligned}&\Big(\frac{e^{-x^2/2}H_{2N}(x)}{2^{2N+1}\pi^{1/2}(2N-1)!}\int_x^\infty e^{-t^2/2}H_{2N-1}(t)\,dt\Big)\Big|_{x=(4N)^{1/2}+2^{-1/2}(2N)^{-1/6}\xi}\\ &\quad\sim\frac{N^{1/6}}{2^{4/3}}\mathrm{Ai}(\xi)\Big\{\int_\xi^\infty\mathrm{Ai}(t)\,dt-\frac{1}{(2N)^{1/3}}\mathrm{Ai}(X)+O(N^{-2/3})\Big\},\end{aligned}\tag{5.6}$$

while it follows from (2.9) that

$$\frac{1}{2}\rho\Big((4N)^{1/2}+2^{-1/2}(2N)^{-1/6}\xi;\mathrm{UE}_{2N}(e^{-x^2})\Big)\ \sim\ \frac{(2N)^{1/6}}{\sqrt{2}}\Big((\mathrm{Ai}'(\xi))^2-\xi(\mathrm{Ai}(\xi))^2+O\Big(\frac{1}{N^{2/3}}\Big)\Big).\tag{5.7}$$

Substituting (5.7) and (5.6) in (5.3) and recalling (5.5) gives the stated result. □

5.2 The LOE and LSE

The soft edge scaling variables for the LOE and LSE are the same as those for the LUE, exhibited in (2.10). The leading correction term to the corresponding soft edge densities are given by the following result.

Proposition 11. *We have*

$$\begin{aligned}&\frac{1}{(2N)^{2/3}}\rho\Big(1+\frac{\xi}{(2N)^{2/3}};\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})\Big)\\ &\quad\sim(\mathrm{Ai}'(\xi))^2-\xi(\mathrm{Ai}(\xi))^2+\frac{1}{2}\mathrm{Ai}(\xi)\Big(1-\int_\xi^\infty\mathrm{Ai}(s)\,ds\Big)\\ &\qquad-\frac{(\alpha-1)}{2(2N)^{1/3}}\Big[\mathrm{Ai}'(\xi)\Big(1-\int_\xi^\infty\mathrm{Ai}(s)\,ds\Big)-(\mathrm{Ai}(\xi))^2\Big]+O(N^{-2/3})\end{aligned}\tag{5.8}$$

and

$$\begin{aligned}\frac{2}{(4N)^{2/3}}\rho\Big(1+\frac{\xi}{(4N)^{2/3}};\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})\Big)\ &\sim\ (\mathrm{Ai}'(\xi))^2-\xi(\mathrm{Ai}(\xi))^2-\frac{1}{2}\mathrm{Ai}(\xi)\int_\xi^\infty\mathrm{Ai}(s)\,ds\\ &+\frac{\alpha+1}{2(4N)^{1/3}}\Big((\mathrm{Ai}(\xi))^2+\mathrm{Ai}'(\xi)\int_\xi^\infty\mathrm{Ai}(t)\,dt\Big)+O(N^{-2/3}).\end{aligned}\tag{5.9}$$

Proof. Consider first the LOE. We rewrite (4.18) as

$$\begin{aligned}\rho(x;\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-x/2}))&=\rho(x;\mathrm{UE}_{N-1}(x^\alpha e^{-x})+\frac{(N-1)!}{4(\alpha+N-2)!}x^{(\alpha-1)/2}e^{-x/2}L_{N-1}^\alpha(x)\\ &\times\Big(2\int_x^\infty L_{N-2}^\alpha(t)t^{(\alpha-1)/2}e^{-t/2}\,dt-\int_0^\infty L_{N-2}^\alpha(t)t^{(\alpha-1)/2}e^{-t/2}\,dt\Big).\end{aligned}\tag{5.10}$$

The asymptotic form of the final integral in (5.9) is known from (4.22). According to (4.21), we seek the asymptotic form of the remaining terms with

$$x = 4N + 2(2N)^{1/3}\xi.$$

For the first integral, use of (3.17) gives

$$\int_{4N+2(2N)^{1/3}\xi}^{\infty} L_{N-2}^{\alpha}(t) t^{(\alpha-1)/2} e^{-t/2}\, dt \\ \sim N^{(\alpha-1)/2}\Big(\int_{\xi}^{\infty} \mathrm{Ai}(s)\, ds - \frac{3-\alpha}{(2N)^{1/3}} \mathrm{Ai}(\xi) + O(N^{-2/3})\Big).$$

The asymptotic form of the term outside the bracketed integrals is given by (3.17) with $p = -1$. For the first term on the right hand side of (5.10), use of (4.19), the analogue of (4.21), and (2.10) shows

$$\rho(4N + 2(2N)^{1/3}\xi; \mathrm{UE}_{N-1}(x^{\alpha}e^{-x})) \\ \sim \frac{1}{2(2N)^{1/3}}\Big((\mathrm{Ai}'(\xi))^2 - \xi(\mathrm{Ai}(\xi))^2 + \frac{\alpha}{2^{1/3}}(\mathrm{Ai}(\xi))^2 \frac{1}{N^{1/3}}\Big) - 2^{-2/3}N^{-2/3}(\mathrm{Ai}(\xi))^2.$$

The asymptotic form of all terms have now been determined, and (5.8) follows.

In relation to the LSE, we use (4.29) to rewrite (4.30) to read

$$\rho(x; \mathrm{SE}_N(x^{\alpha+1}e^{-x})) = \frac{1}{2}\rho(x; \mathrm{UE}_{2N}(x^{\alpha}e^{-x})) \\ + \frac{\Gamma(1+2N)}{4\Gamma(\alpha+2N)} e^{-x/2} x^{(\alpha-1)/2} L_{2N}^{\alpha}(x) \int_x^{\infty} e^{-t/2} t^{(\alpha-1)/2} L_{2N-1}^{\alpha}(t)\, dt. \tag{5.11}$$

A simple change of variables gives

$$\rho\Big(1 + \frac{\xi}{(4N)^{2/3}}; \mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})\Big) = 8N\rho\Big(8N + 2(4N)^{1/3}\xi; \mathrm{SE}_N(x^{\alpha+1}e^{-x})\Big), \tag{5.12}$$

so we seek the asymptotic form of (5.11) with

$$x = 8N + 2(4N)^{1/3}\xi.$$

For this, we make use of (3.17) to deduce

$$\frac{\Gamma(1+2N)}{4\Gamma(\alpha+2N)} e^{-x/2} x^{(\alpha-1)/2} L_{2N}^{\alpha}(x) \int_x^{\infty} e^{-t/2} t^{(\alpha-1)/2} L_{2N-1}^{\alpha}(t)\, dt \\ \sim -\frac{1}{8} 2^{-2/3} N^{-1/3}\Big(\mathrm{Ai}(\xi) - \frac{\alpha+1}{(4N)^{1/3}}\mathrm{Ai}'(\xi)\Big)\Big(\int_{\xi}^{\infty} \mathrm{Ai}(s)\, ds + \frac{\alpha-1}{(4N)^{1/3}}\mathrm{Ai}(\xi) + O(N^{-2/3})\Big).$$

Further, making use of the analogue of (5.12) and recalling (2.10) shows

$$\frac{1}{2}\rho(8N + 2(4N)^{1/3}\xi; \mathrm{UE}_{2N}(x^{\alpha}e^{-x})) \\ \sim \frac{1}{4(4N)^{1/3}}\Big((\mathrm{Ai}'(\xi))^2 - \xi(\mathrm{Ai}(\xi))^2 + \frac{\alpha}{2^{1/3}(2N)^{1/3}}(\mathrm{Ai}(\xi))^2 + O(N^{-2/3})\Big).$$

The asymptotic form of all terms in (5.11) are now determined. After use of (5.12), (5.9) follows. □

Figure 1 provides a numerical comparison of the asymptotic expansion given by (5.9) with the exact result for the LSE density given by (5.11) and (5.12).

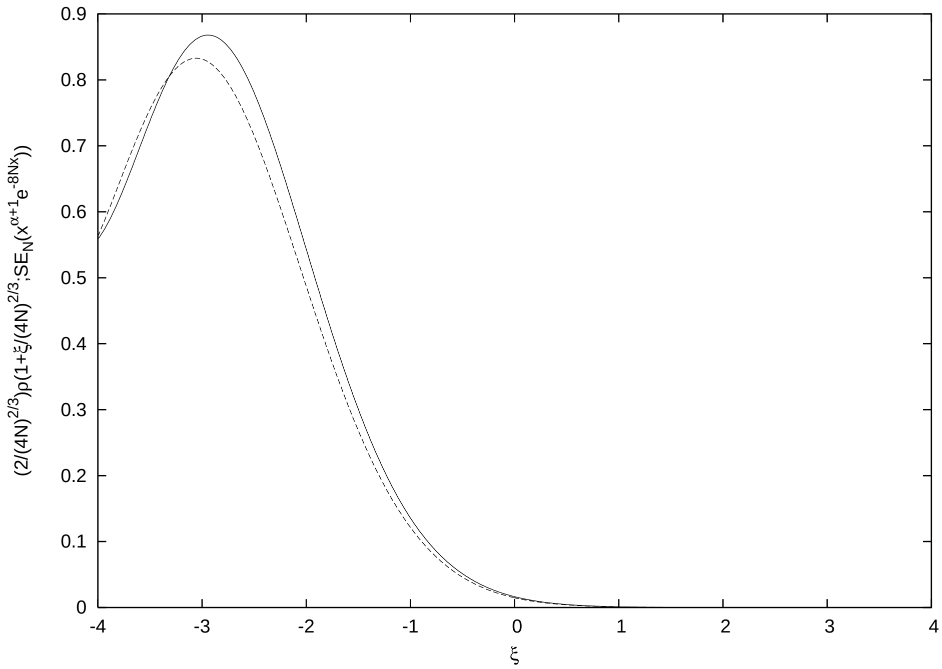

Figure 1: Numerical comparison of the asymptotic expansion (5.9), shown as the dashed line, and the exact result for the LSE density given by (5.11) and (5.12), shown as the solid line, for the eigenvalue density near the soft edge of the LSE with $N = 20$ and $\alpha = 1/2$.

6 Consequences

6.1 Matching the bulk and edge expansions

The GOE and GSE

In Section 2.2 the conjectured matching between the bulk asymptotic expansion expanded about the soft edge, and the asymptotic expansion of the edge density expanded into the bulk, for both the GUE and LUE was given further credence by its confirmation on further terms in the asymptotic expansion. To study the matching in the case of orthogonal and symplectic symmetry, it is therefore convenient to express the corresponding asymptotic expansions in terms of the unitary symmetry expansions. We will consider first the Gaussian cases.

According to (4.3), (4.13), (2.5), (2.12)

$$\frac{1}{N}\rho(X;\mathrm{OE}_N(e^{-Nx^2})) \sim \frac{1}{N}\rho(X;\mathrm{UE}_N(e^{-Nx^2})) + \Big(\frac{\cos[2N\pi P_{\rm W}(X)]}{4\pi(1-X^2)} - \frac{1}{2\pi\sqrt{1-X^2}}\Big)\frac{1}{N}$$
$$+\Big(\frac{1+2X^2}{8\pi(1-X^2)^{5/2}} - \frac{X(21+2X^2)\sin[2N\pi P_{\rm W}(X)]}{48\pi(1-X^2)^{5/2}} - \frac{\cos[2N\pi P_{\rm W}(X)]}{8\pi(1-X^2)^2}\Big)\frac{1}{N^2}; \qquad (6.1)$$

according to (5.1) and (2.9)

$$\frac{1}{2N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{OE}_N(e^{-Nx^2})\Big) \ \sim\ \frac{1}{2N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{UE}_N(e^{-2Nx^2})\Big)$$
$$+\frac{1}{2}\mathrm{Ai}(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(t)\,dt\Big)+\frac{1}{2N^{1/3}}\mathrm{Ai}'(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(t)\,dt\Big)+O\Big(\frac{1}{N^{2/3}}\Big); \quad (6.2)$$

according to (4.17) and (2.5)

$$\frac{1}{N}\rho(X;\mathrm{SE}_N(e^{-4Nx^2})) \ \sim\ \frac{1}{2N}\rho(X;\mathrm{UE}_{2N}(e^{-2Nx^2}))+\frac{\cos[4N\pi P_{\mathrm{W}}(X)]}{8\pi(1-X^2)N}$$
$$\frac{1}{4\pi N\sqrt{1-X^2}}-\Big(\frac{1}{\sqrt{2\pi N}}+\frac{1}{2\pi N}\Big)\frac{\cos[2N\pi P_{\mathrm{W}}(X)+\frac{1}{2}\mathrm{Arcsin}\,X]}{(1-X^2)^{1/4}}+O(N^{-3/2}); \ (6.3)$$

according to (5.4) and (2.9)

$$\frac{1}{(2N)^{2/3}}\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{SE}_N(e^{-4Nx^2})\Big) \ \sim\ \frac{1}{2(2N)^{2/3}}\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{UE}_{2N}(e^{-4Nx^2})\Big)$$
$$-\frac{1}{2}\mathrm{Ai}(\xi)\int_\xi^\infty \mathrm{Ai}(t)\,dt+\frac{(\mathrm{Ai}(\xi))^2}{2(2N)^{1/3}}+O\Big(\frac{1}{N^{2/3}}\Big); \quad (6.4)$$

Substituting (2.11) in (6.1) and (6.3) and expanding as in (2.18) gives

$$\frac{1}{2N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{OE}_N(e^{-Nx^2})\Big) \quad \dot\sim \quad \frac{1}{2N^{2/3}}\rho\Big(1+\frac{\xi}{2N^{2/3}};\mathrm{UE}_N(e^{-2Nx^2})\Big)$$
$$+\frac{3}{16\pi|\xi|^{5/2}}+\frac{23\sin(4|\xi|^{3/2}/3)}{96\pi|\xi|^{5/2}}+\frac{\cos(4|\xi|^{3/2}/3)}{8\pi|\xi|}$$
$$-\Big(\frac{1}{4\pi|\xi|^{1/2}}+\frac{\cos(4|\xi|^{3/2}/3)}{16\pi|\xi|^2}\Big)\frac{1}{N^{1/3}}, \quad (6.5)$$

and

$$\frac{1}{(2N)^{2/3}}\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{SE}_N(e^{-4Nx^2})\Big)\dot\sim\frac{1}{2(2N)^{2/3}}\rho\Big(1+\frac{\xi}{2(2N)^{2/3}};\mathrm{UE}_{2N}(e^{-4Nx^2})\Big)$$
$$-\frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{\sin(\frac{2}{3}|\xi|^{3/2}+\pi/4)}{|\xi|^{1/4}}+\frac{\cos(\frac{4}{3}|\xi|^{3/2})}{8\pi|\xi|}$$
$$+\Big(\frac{1}{4\pi|\xi|^{1/2}}+\frac{|\xi|^{1/4}\sin(\frac{2}{3}|\xi|^{3/2}-\pi/4)}{4\sqrt{\pi}}\Big)\frac{1}{(2N)^{1/3}}+O(N^{-1/2}). \quad (6.6)$$

We now compare (6.5) and (6.6) to the $\xi\to-\infty$ expansion of (6.2) and (6.4) respectively. In relation to the latter, a straightforward calculation using the leading two terms of (2.19) shows that for $|\xi|\to\infty$,

$$\int_{|\xi|}^\infty \mathrm{Ai}(-t)\,dt \ \sim\ \frac{1}{\sqrt{\pi}}\Big(\frac{1}{|\xi|^{3/4}}\cos(\frac{2}{3}|\xi|^{3/2}+\pi/4)+\frac{41}{48}\frac{1}{|\xi|^{9/4}}\sin(\frac{2}{3}|\xi|^{3/2}+\pi/4)\Big).$$

Using this, together with (2.19) itself, we find that (6.5) agrees with the $\xi\to-\infty$ expansion of (6.2) for the first two terms of the $O(1)$ part in (6.5), but the rational factor of $\frac{1}{8}$ should be $\frac{1}{4}$ for the third term. At $O(N^{-1/3})$ agreement is obtained with the first term. Similarly, we find that the the first term at each order in (6.6) agrees with the $\xi\to-\infty$ expansion of (6.6). These results are all consistent with the matching hypothesis.

The LOE and LSE

As in the Gaussian cases, we begin by expressing the densities for the LOE and LSE in terms of the corresponding densities for the LUE.

According to (4.20) and (2.7)

$$\begin{aligned}\frac{1}{N}\rho(X;\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})) &\sim \frac{1}{N}\rho(X;\mathrm{UE}_N(x^{\alpha}e^{-4Nx})) \\ &+\Big(\frac{\cos 2A_{N,\alpha}(X)}{4\pi X(1-X)}-\frac{1}{2\pi(X(1-X))^{1/2}}\Big)\frac{1}{N}; \end{aligned}\tag{6.7}$$

according to (5.8) and (2.10)

$$\begin{aligned}\frac{1}{(2N)^{2/3}}\rho(1+\frac{\xi}{(2N)^{2/3}};\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})) &\sim \frac{1}{(2N)^{2/3}}\rho(1+\frac{\xi}{(2N)^{2/3}};\mathrm{UE}_N(x^{\alpha}e^{-4Nx})) \\ +\frac{1}{2}\mathrm{Ai}(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(s)\,ds\Big)-\frac{(\alpha-1)}{2(2N)^{1/3}}\mathrm{Ai}'(\xi)\Big(1-\int_\xi^\infty \mathrm{Ai}(s)\,ds\Big)&-\frac{(\alpha+1)}{2(2N)^{1/3}}(\mathrm{Ai}(\xi))^2; \end{aligned}\tag{6.8}$$

according to (4.31) and (2.7)

$$\begin{aligned}\frac{1}{N}\rho(X;\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})) &\sim \frac{1}{2N}\rho(X;\mathrm{UE}_{2N}(x^{\alpha}e^{-8Nx})) \\ -\frac{1}{2(\pi N)^{1/2}}\frac{g_{0,2N}^{(L)}(X)}{X^{3/4}(1-X)^{1/4}}&+\frac{\cos 2A_{2N,\alpha}(X)}{8\pi NX(1-X)}+\frac{1}{4\pi N\sqrt{X(1-X)}}; \end{aligned}\tag{6.9}$$

according to (5.9) and (2.10)

$$\begin{aligned}\frac{2}{(4N)^{2/3}}\rho(1+\frac{\xi}{(4N)^{2/3}};\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})) &\sim \frac{1}{(4N)^{2/3}}\rho(1+\frac{\xi}{(4N)^{2/3}};\mathrm{UE}_{2N}(x^{\alpha}e^{-8Nx})) \\ -\frac{1}{2}\mathrm{Ai}(\xi)\int_\xi^\infty \mathrm{Ai}(s)\,ds+\frac{(\alpha+1)}{2(4N)^{1/3}}\mathrm{Ai}'(\xi)\int_\xi^\infty \mathrm{Ai}(t)\,dt&-\frac{(\alpha-1)}{2(4N)^{1/3}}(\mathrm{Ai}'(\xi))^2. \end{aligned}\tag{6.10}$$

Substituting (2.11) in (6.7) and (6.9) and expanding gives

$$\begin{aligned}\frac{1}{(2N)^{2/3}}\rho(1+\frac{\xi}{(2N)^{2/3}};\mathrm{OE}_N(x^{(\alpha-1)/2}e^{-2Nx})) &\dot\sim \frac{1}{(2N)^{2/3}}\rho(1+\frac{\xi}{(2N)^{2/3}};\mathrm{UE}_N(x^{\alpha}e^{-4Nx})) \\ +\frac{\cos(4|\xi|^{3/2}/3)}{4\pi|\xi|}-\frac{(1+\alpha\sin(4|\xi|^{3/2}/3))}{2\pi\sqrt{|\xi|}}\frac{1}{(2N)^{1/3}} \end{aligned}\tag{6.11}$$

and

$$\begin{aligned}\frac{2}{(4N)^{2/3}}\rho(1+\frac{\xi}{(4N)^{2/3}};\mathrm{SE}_N(x^{\alpha+1}e^{-8Nx})) &\dot\sim \frac{1}{(4N)^{2/3}}\rho(1+\frac{\xi}{(4N)^{2/3}};\mathrm{UE}_{2N}(x^{\alpha}e^{-8Nx})) \\ +\frac{\sin(2|\xi|^{3/2}/3-3\pi/4)}{2\sqrt{\pi}|\xi|^{1/4}}+\frac{\cos(4|\xi|^{3/2}/3)}{4\pi|\xi|} \\ +\frac{1}{2(4N)^{1/3}\pi\sqrt{|\xi|}}\Big(1+(1+\alpha)\sqrt{\pi}|\xi|^{3/4}\cos(2|\xi|^{3/2}-3\pi/4)-\frac{\alpha}{2^{1/3}}\sin(4|\xi|^{3/2}/3)\Big) \end{aligned}\tag{6.12}$$

On the other hand let us expand (6.8) and (6.10) for $\xi\to-\infty$. Doing so we find agreement with the first term at each order in (6.11) and (6.12) respectively, as consistent with the matching hypothesis.

6.2 Microscopic delta functions

The results of Sections 4 and 5 tell us the asymptotic expansion of the global density, and the soft edge density. Here we would like to relate these expansions to the result (1.4) and its Laguerre analogue.

Consider first the Gaussian cases. For $|\xi|$ large but otherwise arbitrary, we write

$$\begin{aligned}
&\int_{-\infty}^{\infty} \rho(x; \mathrm{ME}_N(e^{-\beta x^2/2}))a(x)\,dx \\
&= \Big(\int_{R_1} + \int_{R_2}\Big)\rho\Big(X; \mathrm{ME}_N(e^{-N\beta x^2}))\tilde{a}(X)\,dX \\
&= \int_{R_1} \rho(X; \mathrm{ME}_N(e^{-N\beta x^2}))\tilde{a}(X)\,dX \\
&+ \frac{1}{2N^{2/3}}\Big(\int_{-|\xi|}^{\infty} \rho(1 + y/2N^{2/3}; \mathrm{ME}_N(e^{-\beta N x^2}))\tilde{a}(1 + y/2N^{2/3})\,dy \\
&+ \int_{-\infty}^{|\xi|} \rho(1 - y/2N^{2/3}; \mathrm{ME}_N(e^{-\beta N x^2}))\tilde{a}(-1 - y/2N^{2/3})\,dy\Big)
\end{aligned} \tag{6.13}$$

where $\mathrm{ME}_N = \mathrm{OE}_N$, UE_N, SE_N respectively, $R_1 = (-1 + |\xi|/2N^{2/3}, 1 - |\xi|/2N^{2/3})$ and $R_2 = (-\infty, \infty)\backslash R_1$, $\tilde{a}(x) = a(Nx)$ and we have used the fact that ρ is even. Because

$$\rho^{\mathrm{soft}}(y; \mathrm{ME}_N(e^{-\beta N x^2})) := \lim_{N\to\infty} \frac{1}{2N^{2/3}}\rho(1 + y/2N^{2/3}; \mathrm{ME}_N(e^{-\beta N x^2})) \tag{6.14}$$

is an $O(1)$ quantity, we see that to leading order the second and third integrals in (6.13) contribute

$$(\tilde{a}(1) + \tilde{a}(-1))\int_{-|\xi|}^{\infty} \rho^{\mathrm{soft}}(y; \mathrm{ME}_N(e^{-\beta N x^2}))\,dy. \tag{6.15}$$

However, in relation to the first integral on the right hand side of (6.13), we know that terms which are different order in N in the asymptotic expansion of $\rho(X; \mathrm{ME}_N(e^{-N\beta x^2}))$ can contribute at the same order upon the substitution (2.11). Unlike the situation at the edge, the asymptotic expansion of this integral does not therefore correspond directly to the asymptotic of the integrand, leaving us without a method of analysis. Nonetheless some insight into the microscopic origin of the delta functions in (1.4) can be obtained as a consequence of the functional form of (6.14) for $\beta = 1, 4$.

For $\beta = 2$ we read off from (2.9)

$$\rho^{\mathrm{soft}}(y; \mathrm{UE}_N(e^{-2Nx^2})) = (\mathrm{Ai}\,'(y))^2 - y(\mathrm{Ai}(y))^2,$$

while (2.18) tells us that the leading $y \to -\infty$ behavior is $2\sqrt{|y|}/\pi$ so (6.15) diverges for $|\xi| \to \infty$. Because of the result (1.4) it must be that this is exactly canceled by a contribution from the bulk, and thus the edge terms (6.15) cancel.

For $\beta = 1$ and 4 we observe from (5.1) and (5.4) that $\rho^{\mathrm{soft}}(y; \mathrm{UE}_N(e^{-2Nx^2})$ appears as an additive component in the scaled soft edge density, together with a further term in both cases. The further term has the property that it is integrable for $y \to -\infty$. Thinking then of

the decomposition (6.13) for $|\xi|$ large, and ignoring the contribution from the non-integrable additive component, this then suggests that

$$\int_{-\infty}^{\infty}\rho(x;\mathrm{OE}_N(e^{-x^2/2}))a(x)\,dx \sim N\int_{-1}^{1}\rho_{\mathrm{W}}(X)\tilde{a}(X)\,dX - \frac{1}{2\pi}\int_{-1}^{1}\frac{\tilde{a}(X)}{\sqrt{1-X^2}}\,dX$$
$$+(\tilde{a}(1)+\tilde{a}(-1))\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{Ai}(y)\Big(1-\int_{y}^{\infty}\mathrm{Ai}(t)\,dt\Big)dy \tag{6.16}$$

$$\int_{-\infty}^{\infty}\rho(x;\mathrm{SE}_N(e^{-2x^2}))a(x)\,dx \sim N\int_{-1}^{1}\rho_{\mathrm{W}}(X)\tilde{a}(X)\,dX + \frac{1}{4\pi}\int_{-1}^{1}\frac{\tilde{a}(X)}{\sqrt{1-X^2}}\,dX$$
$$-(\tilde{a}(1)+\tilde{a}(-1))\frac{1}{4}\int_{-\infty}^{\infty}\mathrm{Ai}(y)\Big(\int_{y}^{\infty}\mathrm{Ai}(t)\,dt\Big)dy. \tag{6.17}$$

Here the bulk contributions are the leading two non-oscillatory terms exhibited in (4.3) and (4.17) respectively, and the edge contributions are the leading terms in (5.1) and (5.4) respectively, with the component corresponding to the $\beta = 2$ edge deleted. Since

$$\mathrm{Ai}(y)\Big(1-\int_{y}^{\infty}\mathrm{Ai}(t)\,dt\Big) = \frac{1}{2}\frac{d}{dy}\Big(1-\int_{y}^{\infty}\mathrm{Ai}(t)\,dt\Big)^2 \tag{6.18}$$
$$\mathrm{Ai}(y)\int_{y}^{\infty}\mathrm{Ai}(t)\,dt = -\frac{1}{2}\frac{d}{dy}\Big(\int_{y}^{\infty}\mathrm{Ai}(t)\,dt\Big)^2 \tag{6.19}$$

we see from $\int_{-\infty}^{\infty}\mathrm{Ai}(t)\,dt = 1$ that the final integrals in (6.16) and (6.17) are both equal to 1/2, thus reclaiming (1.4).

We consider now the Laguerre analogue of (1.4). Let us introduce the so called chiral matrix ensembles $\mathrm{chME}_N(g(x))$ by the eigenvalue p.d.f.

$$\frac{1}{C}\prod_{l=1}^{N} g(x_l)x_l^{\beta/2}\prod_{1\le j<k\le N}|x_k^2-x_j^2|^{\beta}, \qquad (x_l>0). \tag{6.20}$$

The simple change of variables $x_j^2 \mapsto x_j$ shows

$$\frac{1}{2}\rho(X;\mathrm{chME}_N(x^{2(a+(2-\beta)/4)}e^{-2N\beta x^2})) = X\rho(X^2;\mathrm{ME}_N(x^a e^{-2N\beta x})), \tag{6.21}$$

and thus that the Laguerre ensembles can be viewed as a Gaussian version of the chiral ensemble, generalized by the factor $x^{2(a+(2-\beta)/4)}$. We see from (6.21), (4.20), (2.7) and (4.31) that

$$\rho(X;\mathrm{chOE}_N(x^{a/2}e^{-2Nx^2})) \sim 2\rho_{\mathrm{W}}(X) + \frac{1}{\pi N}\frac{a-1/2}{\sqrt{1-X^2}}$$
$$\rho(X;\mathrm{chUE}_N(x^{a}e^{-4Nx^2})) \sim 2\rho_{\mathrm{W}}(X) + \frac{a}{\pi N\sqrt{1-X^2}}$$
$$\rho(X;\mathrm{chSE}_N(x^{2a}e^{-8Nx^2})) \sim 2\rho_{\mathrm{W}}(X) + \frac{a+1/4}{\pi N\sqrt{1-X^2}}$$

where only non-oscillatory terms are included. In the case $a = 0$ these expansions are precisely the same as for the corresponding Gaussian ensembles (the leading term in the chiral ensembles is $2\rho_{\mathrm{W}}(X)$ rather than $\rho_{\mathrm{W}}(X)$ because $X \in (0,1)$ rather than $(-1,1)$).

At the soft edge $X = 1$ of the chiral ensembles, the fact that the scaled densities are the same as for the Gaussian ensembles tells us that there is a contribution

$$\frac{1}{2N}\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\delta(X-1)$$

to the smoothed density. And, although not the focus of attention of the present work, for the Laguerre and thus chiral ensembles there is an edge effect at $X = 0$ (the so called hard edge [6]) which one expects to give a microscopic contribution

$$-\frac{a}{2N}\delta^{+}(X)$$

(see the next subsection) where $\int_0^\infty f(X)\delta^+(X)\,dX = f(0)$. Consequently we expect the Laguerre analogue of (1.4) to be

$$\begin{aligned}&\rho(X; \mathrm{chME}_N(x^{\beta a/2}e^{-2\beta N x^2}))\\ &\quad\sim 2\rho_{\mathrm{W}}(X) + \frac{a}{\pi N\sqrt{1-X^2}} - \frac{a}{2N}\delta^{+}(X) + \frac{1}{N}\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\Big(\delta(X-1) - \frac{1}{\pi}\frac{1}{\sqrt{1-X^2}}\Big). \end{aligned} \tag{6.22}$$

6.3 Macroscopic balance

In this final subsection, we will show that the results (1.4), (6.22) are consistent with macroscopic considerations.

In a one-component log-gas, to leading order in N the electrostatic potential created by the particle density $\sigma(x)$ must balance the background potential $u(x)$, and thus the equation

$$u(x) + C = \int_{-c}^{c} \sigma(y)\log|x-y|\,dy, \qquad x \in (-c, c) \tag{6.23}$$

where C is *some* constant, must hold. This is to be solved subject to the particle conservation constraint

$$\int_{-c}^{c} \sigma(y)\,dy = 1. \tag{6.24}$$

For example, with $u(x) = x^2$, (6.23) and (6.24) are satisfied with

$$\sigma(y) = \rho_{\mathrm{W}}(y) \tag{6.25}$$

(see e.g. [5]). To the next order in N, fluctuations in the particle density create an electric field and thus a force density which must be balanced for the system to be stable. The balancing force is provided by the gradient of the pressure fluctuation, and leads to the refinement of (6.23) [3]

$$u(x) + C = \int_{-c}^{c} \sigma(y)\log|x-y|\,dy + \frac{1}{N}\Big(\frac{1}{2}-\frac{1}{\beta}\Big)\log\sigma(x), \qquad x \in (-c, c) \tag{6.26}$$

which again is subject to (6.24). With $u(x) = x^2$, setting

$$\sigma(y) = \rho_{\mathrm{W}}(y) + \frac{\mu(y)}{N} \tag{6.27}$$

we see that

$$C = \int_{-c}^{c} \mu(y) \log|x-y| \, dy + \Big(\frac{1}{2} - \frac{1}{\beta}\Big) \log \rho_{\rm W}(x), \qquad x \in (-c,c) \tag{6.28}$$

which must be solved subject to the constraint

$$\int_{-c}^{c} \mu(y) \, dy = 0. \tag{6.29}$$

Differentiating (6.28) gives

$$0 = {\rm PV} \int_{-1}^{1} \frac{\mu(y)}{x-y} \, dy - \Big(\frac{1}{2} - \frac{1}{\beta}\Big) \frac{x}{1-x^2}, \tag{6.30}$$

where PV denotes the principal value. Making use of the fact that

$$ {\rm PV} \int_{-1}^{1} \frac{1}{x-y} \frac{1}{\sqrt{1-y^2}} \, dy = 0, \qquad x \in (-1,1)$$

(see e.g. [15]), we see that (6.30), (6.29) is solved by

$$\mu(y) = \Big(\frac{1}{\beta} - \frac{1}{2}\Big)\Big(\frac{1}{2}\Big(\delta(y-1) + \delta(y+1)\Big) - \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}\Big)$$

in agreement with (1.4).

The chiral ensemble (6.20) is a log-gas confined to $x > 0$ with image charges in $x < 0$. At leading order in N balance of the electric field requires

$$u(x) + C = \int_{0}^{c} \sigma(y) \log|x^2 - y^2| \, dy, \qquad x \in (-c,c), \tag{6.31}$$

subject to the constraint

$$\int_0^c \sigma(y) \, dy = 0.$$

But (6.31) can be written

$$u(x) + C = \int_{-c}^{c} \sigma(y) \log|x-y| \, dy$$

so we have essentially the previous situation. In particular, with $u(x) = 2x^2$, this shows (6.31) is satisfied with

$$\sigma(y) = 2\rho_{\rm W}(y). \tag{6.32}$$

To next order in N the chiral ensembles in (6.22) have $u(x) = 2x^2 - \frac{a}{2N}\log x$, and the generalization of (6.28) reads

$$-\frac{a}{2}\log x + C = \int_{-1}^{1} \mu(y) \log|x-y| \, dy + \Big(\frac{1}{2} - \frac{1}{\beta}\Big) \log \rho_{\rm W}(x).$$

With $\mu(y) \mapsto \mu(y) - \frac{a}{2}\delta(x)$ this is in fact identical to (6.28) and we thus reclaim (6.22).

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